Determinant and Inverse Matrix

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Definition 1. A $n \times n$ square matrix A is **invertible** if there exists a $n \times n$ matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$, where I_n is the identity $n \times n$ matrix. If A^{-1} exists, we say A^{-1} is the inverse matrix of A.

Proposition 2. *If A and B are $n \times n$ matrices, then $AB = I_n \iff BA = I_n$.*

Example 3.

$$\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix} \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$$

So $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$ *is invertible and its inverse is* $\begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}$

Remark 4. If A is invertible, then it follows directly from definition that A^{-1} is also invertible and the inverse of A^{-1} is A.

Proposition 5. *If A, B are $n \times n$ matrices, then:*

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$

It is a natural question to ask if there is some way to tell if a matrix is invertible before looking for its inverse. It turns out that the concept of determinant solve the problem completely.

We will define determinant of a $n \times n$ matrix in a recursive manner.

Definition 6. A is a $n \times n$ square matrix, where $n > 1$. Define the matrix A_{ij} to be the $(n-1) \times (n-1)$ square matrix obtain from A by deleting the i-th row and j-th column.

Example 7. *Let* $A = \begin{bmatrix} 2 & -3 & 1 \\ -5 & 8 & 9 \\ 4 & 7 & -2 \end{bmatrix}$,

then $A_{12} = \begin{bmatrix} -5 & 9 \\ 4 & -2 \end{bmatrix}$, $A_{33} = \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}$

Definition 8. If $A = \{a\}$ is a 1×1 matrix, define the determinant of A to be the number

$$\det(A) = a$$

If $A = \{a_{ij}\}$ is a $n \times n$ $(n > 1)$ square matrix, then define the determinant of A to be the number

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{i1}$$

We now take the case $n = 2$: If $A = \{a_{ij}\}$ is a 2×2 matrix, then bu definition,

$$\det(A) = \sum_{i=1}^{2} (-1)^{i+1} a_{i1} \det(A_{i1}) = a_{11} \det A_{11} - a_{21} \det A_{21} = a_{11}a_{22} - a_{21}a_{12}$$

So we have obtained the formula

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \tag{0.1}$$

We can thus directly apply Formula 0.1 when computing the determinant of a 2×2 matrix.

Example 9.

$$\det \begin{bmatrix} 1 & -5 \\ 4 & 2 \end{bmatrix} = 1 \times 2 - 4 \times (-5) = 22$$

Next let's compute the determinant of a 3×3 matrix:

Example 10.

$$\begin{aligned} \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} &= 1 \times \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 4 \times \det \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} + 7 \times \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix} \\ &= 1 \times (5 \times 9 - 6 \times 8) - 4 \times (2 \times 9 - 3 \times 8) + 7 \times (2 \times 6 - 3 \times 5) \\ &= 1 \times (-3) - 4 \times (-6) + 7 \times (-3) \\ &= 0 \end{aligned}$$

There are some interesting properties regarding determinants:

Proposition 11. *Let A be an $n \times n$ matrix, then:*

1. $\det(I_n) = 1$
2. $\det(A) = \det(A^T)$.
3. *If all entries in some row (column) are* 0, *then* $\det(A) = 0$.
4. *If B is obtained from A by switching two rows (or columns), then* $\det(B) = -\det(A)$.
5. *If B is obtained from A by multiplying some number λ to all the entries in one of the rows (columns), then* $\det(B) = \lambda \det(A)$.
6. *If B is obtained from A by adding a multiple of one row (column) to another, then* $\det(A) = \det(B)$.
7. *If there are two rows (columns) of A such that one is a multiple of another, then* $\det(A) = 0$
8. $\det$ *is multiplicative: for any $n \times n$ matrices A and B,*

$$\det(AB) = \det(A)\det(B)$$

Example 12. *By the above propositions, we see that* $\det \begin{bmatrix} 1 & 1 & 3 \\ 4 & 4 & 6 \\ 7 & 7 & 9 \end{bmatrix} = 0$. *Then we add* 1 *times the third column to the second column,* $\det \begin{bmatrix} 1 & 4 & 3 \\ 4 & 10 & 6 \\ 7 & 16 & 9 \end{bmatrix} = 0$. *Next we multiply the second column by* $\frac{1}{2}$, *we get* $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \frac{1}{2} \times 0 = 0$, *which agrees with the previous example.*

Remark 13. The above propositions implies we can compute determinant in a more general way:

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) = \sum_{j=1}^{n} (-1)^{i+j} \det(A_{ij})$$

An important application of determinant is that it can be used to test if a square matrix is invertible or not:

Theorem 14. *A square matrix A is invertible if and only if* $\det(A) \neq 0$.

Example 15. *We have shown that* $\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 0$, *so* $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ *is not invertible.*

Example 16. *A 2×2 matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is invertible if and only if*

$$\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc \neq 0$$

The method of Gaussian Elimination can be applied to computing the inverse of a given matrix.

Theorem 17. *A is an invertible $n \times n$ matrix. If we form the matrix*

$$\left[\ A \mid I_n\ \right] = \left[\begin{array}{ccc|cccc} a_{11} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \multicolumn{7}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ a_{n1} & \dots & a_{nn} & 0 & 0 & \dots & 1 \end{array} \right]$$

and apply the method of Gaussian Elimination, we will obtain

$$\left[\ I_n \mid A^{-1}\ \right]$$

Example 18. *Find the inverse of*

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & 0 & -3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array} \right]$$

So we find the inverse of the given matrix is

$$\begin{bmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

When an invertible matrix is 2×2, there is a formula for its inverse:

Proposition 19. *If* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is an invertible matrix, then its inverse is*

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 20. *Find the inverse of* $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{1 \times 4 - 2 \times 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

Proposition 21. *For a $n \times n$ diagonal matrix $A = (a_{ij})$ (i.e. $a_{ij} = 0$ for any $i \neq j$), its determinant is*

$$\det(A) = a_{11}a_{22}...a_{nn}$$

Example 22.

$$\det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix} = 2 \times 3 \times (-4) = -24$$